

$y = 4\sqrt{x}$ is a particular solution of the non-homogeneous differential equation $9x^2y'' - 3xy' + 4y = \sqrt{x}$.

SCORE: ____ / 15 PTS

Solve the initial value problem $9x^2y'' - 3xy' + 4y = 2\sqrt{x}$, $y(1) = -7$, $y'(1) = 5$.

$$\textcircled{2} \quad 9r^2 - 12r + 4 = 0$$

$$(3r - 2)^2 = 0$$

$$\textcircled{1} \quad r = \frac{2}{3}, \frac{2}{3}$$

$$y = 2(4\sqrt{x}) + A x^{\frac{2}{3}} + B x^{\frac{2}{3}} \ln x$$

$$\textcircled{2} = 8\sqrt{x} + \textcircled{2} A x^{\frac{2}{3}} + B x^{\frac{2}{3}} \ln x \textcircled{2}$$

$$y(1) = 8 + A = -7 \rightarrow A = -15 \textcircled{1}$$

$$y' = 4x^{-\frac{1}{2}} + \frac{2}{3} A x^{-\frac{1}{3}} + \frac{2}{3} B x^{-\frac{1}{3}} \ln x + B x^{-\frac{1}{3}} \textcircled{2}$$

$$y'(1) = 4 + \frac{2}{3}(-15) + B = 5$$

$$B = 11 \textcircled{2}$$

$$y = 8\sqrt{x} - 15x^{\frac{2}{3}} + 11x^{\frac{2}{3}} \ln x \textcircled{1}$$

$y = e^{-2x}$ is a solution of the homogeneous differential equation $(2x+1)y'' + 4xy' - 4y = 0$.

SCORE: ____ / 40 PTS

Find the general solution of the non-homogeneous differential equation $(2x+1)y'' + 4xy' - 4y = 2e^{-2x}(2x+1)^2$.

$$y_2 = ve^{-2x}$$

$$y_2' = v'e^{-2x} - 2ve^{-2x}$$

$$y_2'' = v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}$$

$$e^{-2x} [v''(2x+1) + v'(-4(2x+1)) + v(4(2x+1))] \quad (3)$$
$$+ v'(4x) + v(-2(4x)) \quad (2)$$
$$+ v(-4) \quad (1) = 0$$

$$e^{-2x} [v''(2x+1) + v'(-4x-4)] = 0 \quad (2)$$

$$\text{LET } u = v'$$

$$u'(2x+1) + u(-4x-4) = 0$$

$$\int \frac{1}{u} du = \int \frac{4x+4}{2x+1} dx \quad (2)$$

$$= \int \left(2 + \frac{2}{2x+1}\right) dx \quad (2)$$

$$\ln|u| = 2x + \ln|2x+1| \quad (2)$$

$$v' = u = (2x+1)e^{2x} \quad (2)$$

$$v = xe^{2x} \quad (2)$$

$$y_2 = x \quad (2)$$

$$\begin{array}{r} 2x+1 \\ 2 \\ 0 \end{array} \begin{array}{l} \times e^{2x} \\ \times \frac{1}{2}e^{2x} \\ \times \frac{1}{4}e^{2x} \end{array}$$

$$W = \begin{vmatrix} e^{-2x} & x \\ -2e^{-2x} & 1 \end{vmatrix} = (1+2x)e^{-2x} \quad (3)$$

$$g = \frac{2e^{-2x}(2x+1)^2}{2x+1} = 2e^{-2x}(2x+1) \quad (3)$$

$$y_p = -e^{-2x} \int \frac{2e^{2x}(2x+1)x}{(1+2x)e^{2x}} dx + x \int \frac{2e^{-2x}(2x+1)e^{-2x}}{(1+2x)e^{-2x}} dx \quad (2)$$

$$= -e^{-2x} \int 2x dx + x \int 2e^{-2x} dx$$

$$= \frac{3}{2}x^2e^{-2x} - xe^{-2x} \quad (3)$$

$$y = \frac{3}{2}(x^2+x)e^{-2x} + Ae^{-2x} + Bx \quad (2)$$

Find the general solution of $y''' - y'' + y' - y = x + 4\cos x$.

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$$r^3 - r^2 + r - 1 = 0$$

$$r^2(r-1) + (r-1) = 0$$

$$(r^2+1)(r-1) = 0 \quad (2)$$

$$r = \pm i, 1 \quad (1)$$

$$y_h = A\cos x + B\sin x + Ce^x \quad (2)$$

$$y_p = Dx\cos x + Fx\sin x + Gx + H \quad (2)$$

$$y_p' = D\cos x - Dx\sin x + G + Fx\cos x + F\sin x$$

$$= (Fx+D)\cos x + (-Dx+F)\sin x + G \quad (4)$$

$$y_p'' = F\cos x + (-Fx-D)\sin x$$

$$+ (-Dx+F)\cos x - D\sin x$$

$$= (-Dx+2F)\cos x + (-Fx-2D)\sin x \quad (4)$$

$$y_p''' = -D\cos x + (Dx-2F)\sin x$$

$$+ (-Fx-2D)\cos x - F\sin x$$

$$= (-Fx-3D)\cos x + (Dx-3F)\sin x \quad (4)$$

$$y_p''' - y_p'' + y_p' - y_p = (-Fx-3D+Dx-2F+Fx+D-Dx)\cos x \\ + (Dx-3F+Fx+2D-Dx+F-Fx)\sin x \\ + G - Gx - H$$

$$= (-2D-2F)\cos x + (-2F+2D)\sin x - Gx + G - H \quad (4)$$

$$-2D-2F=4$$

$$2D-2F=0$$

$$-4F=4 \rightarrow F=-1 \quad (1\frac{1}{2})$$

$$D=-1 \quad (1\frac{1}{2})$$

$$-G=1 \rightarrow G=-1 \quad (1\frac{1}{2})$$

$$G-H=0 \quad H=-1 \quad (1\frac{1}{2})$$

$$y = -x\cos x - x\sin x - x - 1 + A\cos x + B\sin x + Ce^x \quad (2)$$

$$(2D-4)(x) - (D-2)(y) = 4e^{-2t}$$

$$(D+7)(x) + (2D-1)(y) = -5e^{-2t}$$

$$\begin{aligned} \textcircled{1} \left\{ \begin{aligned} \textcircled{4} (D+7)(2D-4)(x) - (D+7)(D-2)(y) &= -8e^{-2t} + 28e^{-2t} = 20e^{-2t} \textcircled{2} \\ (2D-4)(D+7)(x) + (2D-4)(2D-1)(y) &= 20e^{-2t} + 20e^{-2t} = 40e^{-2t} \textcircled{2} \end{aligned} \right. \end{aligned}$$

$$\textcircled{4} (5D^2 - 5D - 10)(y) = 20e^{-2t} \textcircled{2}$$

$$(D^2 - D - 2)(y) = 4e^{-2t}$$

$$r^2 - r - 2 = 0 \rightarrow r = 2, -1 \rightarrow y_h = Ae^{2t} + Be^{-t}$$

$$y_p = Ke^{-2t} \textcircled{2}$$

$$y_p' = -2Ke^{-2t}$$

$$y_p'' = 4Ke^{-2t}$$

$$y_p'' - y_p' - 2y_p = e^{-2t}(4K + 2K - 2K) = 4Ke^{-2t} = 4e^{-2t} \textcircled{4}$$

$$K = 1$$

$$y = Ae^{2t} + Be^{-t} + e^{-2t} \textcircled{2}$$

$$\textcircled{1} \left\{ \begin{aligned} \textcircled{4} (2D-1)(2D-4)(x) - (2D-1)(D-2)(y) &= -16e^{-2t} - 4e^{-2t} = -20e^{-2t} \textcircled{2} \\ (D-2)(D+7)(x) + (D-2)(2D-1)(y) &= 10e^{-2t} + 10e^{-2t} = 20e^{-2t} \textcircled{2} \end{aligned} \right.$$

$$\textcircled{4} (5D^2 - 5D - 10)(x) = 0$$

$$x = Ce^{2t} + De^{-t} \textcircled{3}$$

$$(D+7)(x) + (2D-1)(y)$$

$$= 2Ce^{2t} - De^{-t}$$

$$+ 7Ce^{2t} + 7De^{-t}$$

$$+ 4Ae^{2t} - 2Be^{-t} - 4e^{-2t}$$

$$- Ae^{2t} - Be^{-t} - e^{-2t}$$

$$= (9C+3A)e^{2t} + (6D-3B)e^{-t} - 5e^{-2t} = -5e^{-2t} \textcircled{6}$$

$$9C+3A=0$$

$$6D-3B=0$$

$$A = -3C \textcircled{3}$$

$$B = 2D \textcircled{3}$$

$$\textcircled{3} \begin{cases} x = Ce^{2t} + De^{-t} \\ y = -3Ce^{2t} + 2De^{-t} + e^{-2t} \end{cases}$$